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GALERKIN STRESS FUNCTIONS FOR NON-LOCAL THEORIES OF ELASTICITY†

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INTRODUCTION

IN THE classical theory of elasticity, the potential energy density is a function of the strain whereas in the nonlocal theories the energy is a function of the strain and gradients of the strain, or parts thereof such as the rotation gradient which leads to couple stresses. For the classical theory, Galerkin [1] exhibited a complete solution of the displacement equation of equilibrium in terms of a single vector function satisfying an equation of higher order but of simpler and more tractable form. In the present paper, after a review of the solutions for the classical and rotation gradient cases, the analogous solutions are exhibited for the non-local equations of higher and higher orders up to Cauchy's [2] equation of infinite order for his asymptotic theory of an elastic medium with long range interactions between molecules in a periodic structure.

CLASSICAL THEORY

The displacement equation of equilibrium in the classical, linear theory of elasticity, for the isotropic case, is

$$k\nabla\nabla\cdot\mathbf{u}-\nabla\times\nabla\times\mathbf{u}+\mathbf{f} = 0. \tag{1}$$

In (1) $\mathbf{u}$ is the displacement, $k = (\lambda+2\mu)/\mu$ where λ is Lamé's constant and μ is the shear modulus, $\mathbf{f}$ is the ratio of the body force density to μ and ∇, $\nabla\cdot$, $\nabla\times$ are the gradient, divergence and curl operators, respectively.

Galerkin's solution of (1) may be obtained as follows. In a region V, let x, y, z be the coordinates of a field point, P, and ξ, η, ζ be the coordinates of a source point Q. Let

$$r = (x^2+y^2+z^2)^{1/2},\ r' = (\xi^2+\eta^2+\zeta^2)^{1/2},\ r_1 = [(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{1/2}.$$

Suppose $\mathbf{u}$ is *any* (sufficiently smooth) vector function and define a vector function $\mathbf{H}$ and a scalar function φ by

$$4\pi\mathbf{H} = -\int_V r_1^{-1}\mathbf{u}_Q\,\mathrm{d}V_Q, \qquad 4\pi\varphi = \int_V r_1^{-1}\mathbf{r}'\cdot\nabla_Q^2\mathbf{H}_Q\,\mathrm{d}V_Q.$$

Then

$$\nabla^2\mathbf{H} = \mathbf{u}, \qquad \nabla^2\varphi = -\mathbf{r}\cdot\nabla^2\mathbf{H}, \tag{2}$$

where ∇^2 is Laplace's operator.

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Now, define the Galerkin function by

$$\mathbf{G} = \mathbf{H}+\tfrac{1}{2}(k-1)\nabla(\mathbf{r}\cdot\mathbf{H}+\varphi). \tag{3}$$

Then

$$\nabla\nabla\cdot\mathbf{G} = \nabla\nabla\cdot\mathbf{H}+\tfrac{1}{2}(k-1)\nabla(\mathbf{r}\cdot\nabla^2\mathbf{H}+2\nabla\cdot\mathbf{H}+\nabla^2\varphi)$$

so that, from the second of (2),

$$\nabla\nabla\cdot\mathbf{G} = k\nabla\nabla\cdot\mathbf{H}. \tag{4}$$

Also, from (3),

$$\nabla\times\nabla\times\mathbf{G} = \nabla\times\nabla\times\mathbf{H} = \nabla\nabla\cdot\mathbf{H}-\nabla^2\mathbf{H}.$$

This becomes, from (4) and the first of (2),

$$\nabla\times\nabla\times\mathbf{G} = k^{-1}\nabla\nabla\cdot\mathbf{G}-\mathbf{u}.$$

Thus, *any* sufficiently smooth vector function $\mathbf{u}$ may be represented by

$$\mathbf{u} = k^{-1}\nabla\nabla\cdot\mathbf{G}-\nabla\times\nabla\times\mathbf{G}. \tag{5}$$

Suppose, now, that $\mathbf{u}$ is a solution of (1). To find the resulting equation governing $\mathbf{G}$, first note that, from (5),

$$\nabla\nabla\cdot\mathbf{u} = k^{-1}\nabla^2\nabla\nabla\cdot\mathbf{G},$$

$$\nabla\times\nabla\times\mathbf{u} = -\nabla\times\nabla\times\nabla\times\nabla\times\mathbf{G} = \nabla\times\nabla\times\nabla^2\mathbf{G} = \nabla^2\nabla\nabla\cdot\mathbf{G}-\nabla^4\mathbf{G},$$

where $\nabla^4 = \nabla^2\nabla^2$. Then upon substituting (5) in (1), we find

$$\nabla^4\mathbf{G} = -\mathbf{f}. \tag{6}$$

Accordingly, Galerkin's complete solution of (1) is given by (5), provided the Galerkin function $\mathbf{G}$ satisfies (6).

ROTATION GRADIENT

The theory of elasticity in which the potential energy density is a function of the strain and the rotation gradient was given by Aero and Kuvshinskii [3], Grioli [4], Rajagopal [5] and Truesdell and Toupin [6]. The displacement equation of equilibrium for the centrosymmetric, isotropic case is

$$k\nabla\nabla\cdot\mathbf{u}-(1-l_2^2\nabla^2)\nabla\times\nabla\times\mathbf{u}+\mathbf{f} = 0, \tag{7}$$

where l_2 is a material constant.

To find the Galerkin function for (7), first define $\mathbf{H}$ and φ, in this case, by

$$4\pi\mathbf{H} = -\int_V r_1^{-1}(1-\mathrm{e}^{-r_1/l_2})\mathbf{u}_Q\,\mathrm{d}V_Q, \qquad 4\pi\varphi = \int_V r_1^{-1}\mathbf{r}'\cdot(1-l_2^2\nabla_Q^2)\nabla_Q^2\mathbf{H}_Q\,\mathrm{d}V_Q.$$

Then [7, equation (10.12)],

$$(1-l_2^2\nabla^2)\nabla^2\mathbf{H} = \mathbf{u}, \qquad \nabla^2\varphi = -\mathbf{r}\cdot(1-l_2^2\nabla^2)\nabla^2\mathbf{H}. \tag{8}$$

Now, define the Galerkin function, in this case, by

$$\mathbf{G} = (1-l_2^2\nabla^2)\mathbf{H}+\tfrac{1}{2}(k-1)\nabla[\mathbf{r}\cdot(1-l^2\nabla^2)\mathbf{H}+\varphi],$$

where

$$l^2 = l_2^2/(1-k).$$

Then, using the second of (8), we find

$$\nabla\nabla\cdot\mathbf{G} = k\nabla\nabla\cdot\mathbf{H}. \tag{9}$$

Also, taking into account (9) and the first of (8), we find

$$\mathbf{u} = k^{-1}(1-l_2^2\nabla^2)\nabla\nabla\cdot\mathbf{G}-\nabla\times\nabla\times\mathbf{G}. \tag{10}$$

To find the equation governing **G**, substitute (10) in (7). The result is

$$(1-l_2^2\nabla^2)\nabla^4\mathbf{G} = -\mathbf{f}. \tag{11}$$

The solution (10) and (11) was given by Mindlin and Tiersten [7, equations (11.24) and (11.25)]. An alternative proof of completeness was given by Doyle [8].

FIRST STRAIN GRADIENT

The theory of elasticity in which the potential energy density is a function of the strain and the first gradient of the strain was given by Toupin [9]. The displacement equation of equilibrium for the centrosymmetric, isotropic case is [10, equation (13.1)]

$$k(1-l_1^2\nabla^2)\nabla\nabla\cdot\mathbf{u}-(1-l_2^2\nabla^2)\nabla\times\nabla\times\mathbf{u}+\mathbf{f} = 0, \tag{12}$$

where l_1 and l_2 are constants. Define **H** and φ, now, as

$$4\pi\mathbf{H} = -\int_V r_1^{-1}[1+(l_2^2\,\mathrm{e}^{-r_1/l_2}-l_1^2\,\mathrm{e}^{-r_1/l_1})/(l_1^2-l_2^2)]\mathbf{u}_Q\,\mathrm{d}V_Q,$$

$$4\pi\varphi = \int_V r_1^{-1}\mathbf{r}'\cdot(1-l^2\nabla_Q^2)\nabla_Q^2\mathbf{H}_Q\,\mathrm{d}V_Q,$$

where

$$l^2 = (kl_1^2-l_2^2)/(k-1). \tag{13}$$

Then [11, equation (67)]

$$(1-l_1^2\nabla^2)(1-l_2^2\nabla^2)\nabla^2\mathbf{H} = \mathbf{u}, \qquad \nabla^2\varphi = -\mathbf{r}\cdot(1-l^2\nabla^2)\nabla^2\mathbf{H}. \tag{14}$$

Next, define the Galerkin function for this case by

$$\mathbf{G} = (1-l_2^2\nabla^2)\mathbf{H}+\tfrac{1}{2}(k-1)\nabla[\mathbf{r}\cdot(1-l^2\nabla^2)\mathbf{H}+\varphi].$$

Then, employing (13) and the second of (14), we find

$$\nabla\nabla\cdot\mathbf{G} = k(1-l_1^2\nabla^2)\nabla\nabla\cdot\mathbf{H}. \tag{15}$$

Also,

$$\nabla\times\nabla\times\mathbf{G} = (1-l_2^2\nabla^2)\nabla\times\nabla\times\mathbf{H} = (1-l_2^2\nabla^2)(\nabla\nabla\cdot\mathbf{H}-\nabla^2\mathbf{H}). \tag{16}$$

From (15) and (16), and employing the first of (14), we find

$$\mathbf{u} = k^{-1}(1-l_2^2)\nabla\nabla\cdot\mathbf{G}-(1-l_1^2\nabla^2)\nabla\times\nabla\times\mathbf{G}.$$

Finally, upon substituting this representation in the equation of equilibrium (12), we find the equation governing the Galerkin function for the strain gradient theory:

$$(1-l_1^2\nabla^2)(1-l_2^2\nabla^2)\nabla^4\mathbf{G} = -\mathbf{f}.$$

SECOND AND HIGHER GRADIENTS OF STRAIN

When the potential energy density is a function of the strain and the first and second gradients of the strain, the displacement equation of equilibrium for the centrosymmetric isotropic case is [11, equation (25)]

$$k(1-l_{11}^2\nabla^2)(1-l_{12}^2\nabla^2)\nabla\nabla\cdot\mathbf{u}-(1-l_{21}^2\nabla^2)(1-l_{22}^2\nabla^2)\nabla\times\nabla\times\mathbf{u}+\mathbf{f} = 0, \qquad (17)$$

where l_{ij} are constants. The pattern of representation having been established in the preceding sections, it may be inferred that any solution of (17) can be expressed in terms of a Galerkin function, $\mathbf{G}$, according to

$$\mathbf{u} = k(1-l_{21}^2\nabla^2)(1-l_{22}^2\nabla^2)\nabla\nabla\cdot\mathbf{G}-(1-l_{11}^2\nabla^2)(1-l_{12}^2\nabla^2)\nabla\times\nabla\times\mathbf{G}. \qquad (18)$$

Upon substituting (18) into (17) we find the following equation to be satisfied by the Galerkin stress function for the second strain gradient theory:

$$(1-l_{11}^2\nabla^2)(1-l_{12}^2\nabla^2)(1-l_{21}^2\nabla^2)(1-l_{22}^2\nabla^2)\nabla^4\mathbf{G} = -\mathbf{f}.$$

Similarly, if the potential energy density is a function of the strain and all gradients of strain up to and including the nth, it may be inferred that the equation of equilibrium, for the centrosymmetric, isotropic case, would be

$$\begin{aligned} &k[(1-l_{11}^2\nabla^2)(1-l_{12}^2\nabla^2)\dots(1-l_{1n}^2\nabla^2)]\nabla\nabla\cdot\mathbf{u} \\ &\quad -[(1-l_{21}^2\nabla^2)(1-l_{22}^2\nabla^2)\dots(1-l_{2n}^2\nabla^2)]\nabla\times\nabla\times\mathbf{u}+\mathbf{f} = 0 \end{aligned} \qquad (19)$$

and the complete representation of the displacement in terms of the Galerkin function would be

$$\begin{aligned} \mathbf{u} = {}&k^{-1}[(1-l_{21}^2\nabla^2)(1-l_{22}^2\nabla^2)\dots(1-l_{2n}^2\nabla^2)]\nabla\nabla\cdot\mathbf{G} \\ &-[(1-l_{11}^2\nabla^2)(1-l_{12}^2\nabla^2)\dots(1-l_{1n}^2\nabla^2)]\nabla\times\nabla\times\mathbf{G}, \end{aligned}$$

where $\mathbf{G}$ satisfies the equation

$$[(1-l_{21}^2\nabla^2)(1-l_{22}^2\nabla^2)\dots(1-l_{2n}^2\nabla^2)][(1-l_{11}^2\nabla^2)(1-l_{12}^2\nabla^2)\dots(1-l_{1n}^2\nabla^2)]\nabla^4\mathbf{G} = -\mathbf{f}.$$

The equation of equilibrium (19) may be written in the form

$$(E_1\nabla^2+E_2\nabla^4+\cdots E_{n+1}\nabla^{2(n+1)})\mathbf{u}+(F_1+F_2\nabla^2+\cdots F_n\nabla^{2n})\nabla\nabla\cdot\mathbf{u}+\mathbf{f} = 0,$$

where $E_1, E_2\dots$ and $F_1, F_2\dots$ are constants. This equation is to be compared with Cauchy's asymptotic equation for an elastic medium with long range interactions between molecules in a periodic structure. In the case of equilibrium of a centrosymmetric, isotropic material, Cauchy's equation is [2]

$$E\mathbf{u}+F\nabla\nabla\cdot\mathbf{u}+\mathbf{f} = 0,$$

where E and F are entire functions of ∇^2. Upon expanding E and F in series of powers of ∇^2, we see that the nth gradient theory comprises the first n terms of Cauchy's theory.

Finally, it may be observed that the Galerkin functions for the gradient theories of various orders are all governed by equations of the form

$$[(1-a_1\nabla^2)(1-a_2\nabla^2)\ldots(1-a_n\nabla^2)]\nabla^4\mathbf{G} = -\mathbf{f}.$$

Since the differential operator, of order $2n+4$, is a product of the double Laplacian and n second order linear operators, the Galerkin function may be resolved into the sum of a particular integral, a biharmonic function, a harmonic function and n functions satisfying homogeneous second order equations:

$$\mathbf{G} = \mathbf{G}_p+\mathbf{G}''+\mathbf{G}'+\sum_{i=1}^{i=n}\mathbf{G}_i,$$

where

$$\nabla^4\mathbf{G}'' = 0, \qquad \nabla^2\mathbf{G}' = 0, \qquad (1-a_i\nabla^2)\mathbf{G}_i = 0, \qquad i = 1,2\ldots n,$$

and G_p is the particular integral.

REFERENCES

[1] B. G. Galerkin, *C. r. hebd. Séanc. Acad. Sci., Paris* **190**, 1047 (1930).
[2] A. L. Cauchy, *C. r. hebd. Séanc. Acad. Sci., Paris* **32**, 323 (1851).
[3] E. L. Aero and E. V. Kuvshinskii, *Fiz. tverd. Tela* **2**, 1399 (1960).
[4] G. Grioli, *Annali. Mat. pura appl.* **50**, 389 (1960).
[5] E. S. Rajagopal, *Ann. Phys.* **6**, 192 (1960).
[6] C. A. Truesdell and R. A. Toupin, *Encyclopedia of Physics*, Vol. III/1. Springer (1960).
[7] R. D. Mindlin and H. F. Tiersten, *Arch. ration. Mech. Analysis* **11**, 415 (1962).
[8] J. M. Doyle, *J. appl. Mech.* **31**, 728 (1964).
[9] R. A. Toupin, *Arch. ration. Mech. Analysis* **11**, 385 (1962).
[10] R. D. Mindlin, *Arch. ration. Mech. Analysis* **16**, 51 (1964).
[11] R. D. Mindlin, *Int. J. Solids Struct.* **1**, 917 (1965).

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